Sirindhorn International Institute of Technology

Thammasat University

School of Information, Computer and Communication Technology

ECS315 2018/1 Part I.2 Dr.Prapun

4 Enumeration / Combinatorics / Counting

There are many probability problems, especially those concerned with gambling, that can ultimately be reduced to questions about cardinalities of various sets. ***Combinatorics*** is the *study of systematic counting methods*, which we will be using to find the cardinalities of various sets that arise in probability.

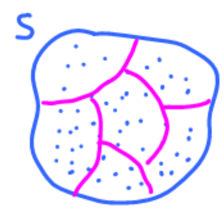
4.1 Four Principles

addition ← non-overlapping cases

multiplication ← steps, tree-diagram

subtraction

division



4.1. ***Addition Principle (Rule of sum)***:

- When there are m **cases** such that the ith case has n_i options, for $i = 1, \ldots, m$, and no two of the cases have any options in common, the total number of options is $n_1 + n_2 + \cdots + n_m$.
- In set-theoretic terms, suppose that a finite set S can be partitioned[5] into (pairwise disjoint parts) $S_1, S_2, \ldots, S_m$. Then,

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

[5] The art of applying the addition principle is to partition the set S to be counted into "manageable parts"; that is, parts which we can readily count. But this statement needs to be qualified. If we partition S into too many parts, then we may have defeated ourselves. For instance, if we partition S into parts each containing only one element, then applying the

In words, "if you can count the number of elements in all of the parts of a partition of S, then $|S|$ is simply the sum of the number of elements in all the parts".

Example 4.2. We may find the number of people living in a country by adding up the number from each province/state.

Example 4.3. [1, p 28] Suppose we wish to find the number of different courses offered by SIIT. We partition the courses according to the department in which they are listed. Provided there is no cross-listing (cross-listing occurs when the same course is listed by more than one department), the number of courses offered by SIIT equals the sum of the number of courses offered by each department.

Example 4.4. [1, p 28] A student wishes to take either a mathematics course or a biology course, but not both. If there are four mathematics courses and three biology courses for which the student has the necessary prerequisites, then the student can choose a course to take in $4 + 3 = 7$ ways.

Example 4.5. Let A, B, and C be finite sets. How many triples are there of the form (a,b,c), where $a \in A$, $b \in B$, $c \in C$? there are $|C|$ choices for c

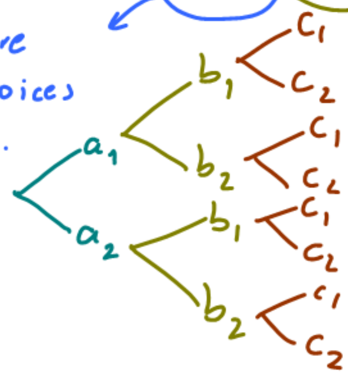
Ex.

There are $|A|$ choices for a.

$A = \{a_1, a_2\}$

$B = \{b_1, b_2\}$

$C = \{c_1, c_2\}$



there are $|B|$ choices for b

The total number of triples is $|A| \times |B| \times |C|$ $= |A \times B \times C|$

4.6. Tree diagrams: When a set can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree. Each of the ways of completing the second step can be represented as n_2 branches starting from

addition principle is the same as counting the number of parts, and this is basically the same as listing all the objects of S. Thus, a more appropriate description is that the art of applying the addition principle is to partition the set S into not too many manageable parts.[1, p 28]

the ends of the original branches, and so forth. The size of the set then equals the number of branches in the last level of the tree, and this quantity equals

$$n_1 \times n_2 \times \cdots$$

4.7. *Multiplication Principle (Rule of product)*:

- When a procedure/operation can be broken down into m **steps**,
  such that there are n_1 options for step 1,
  and such that after the completion of step $i-1$ $(i = 2, \dots, m)$ there are n_i options for step i (for each way of completing step $i-1$),
  the number of ways of performing the procedure is $n_1 n_2 \cdots n_m$.
- In set-theoretic terms, if sets $S_1, S_2, \dots, S_m$ are finite, then $|S_1 \times S_2 \times \cdots \times S_m| = |S_1| \times |S_2| \times \cdots \times |S_m|$.
- For m finite sets $A_1, A_2, \dots, A_m$, there are $|A_1| \times |A_2| \times \cdots \times |A_m|$ m-tuples of the form $(a_1, a_2, \dots, a_m)$ where each $a_i \in A_i$.

Example 4.8. Suppose that a deli offers three kinds of bread, three kinds of cheese, four kinds of meat, and two kinds of mustard. How many different meat and cheese sandwiches can you make?

First choose the bread. For each choice of bread, you then have three choices of cheese, which gives a total of $3 \times 3 = 9$ bread/cheese combinations (rye/swiss, rye/provolone, rye/cheddar, wheat/swiss, wheat/provolone ... you get the idea). Then choose among the four kinds of meat, and finally between the two types of mustard or no mustard at all. You get a total of $3 \times 3 \times 4 \times 3 = 108$ different sandwiches.

Suppose that you also have the choice of adding lettuce, tomato, or onion in any combination you want. This choice gives another 2 x 2 x $2 = 8$ combinations (you have the choice "yes" or "no" three times) to combine with the previous 108, so the total is now $108 \times 8 = 864$.

That was the multiplication principle. In each step you have several choices, and to get the total number of combinations, multiply. It is fascinating how quickly the number of combinations grow. Just add one more type of bread, cheese, and meat, respectively, and the number of sandwiches becomes 1,920. It would take years to try them all for lunch. [17, p 33]

Example 4.9 (Slides)**.** In 1961, Raymond Queneau, a French poet and novelist, wrote a book called *One Hundred Thousand Billion Poems*. The book has ten pages, and each page contains a sonnet, which has 14 lines. There are cuts between the lines so that each line can be turned separately, and because all lines have the same rhyme scheme and rhyme sounds, any such combination gives a readable sonnet. The number of sonnets that can be obtained in this way is thus 10^{14} which is indeed a hundred thousand billion. Somebody has calculated that it would take about 200 million years of nonstop reading to get through them all. [17, p 34]

10^2 10^3 10^9

Example 4.10. There are 2^n binary strings/sequences of length n.

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{n \text{ steps for } n \text{ positions}}\ \_$$

Example 4.11. For a finite set A, the cardinality of its power set of A 2^A is

$$|2^A| = 2^{|A|}.$$

the set of all subsets of A (including the empty set and the set A itself).

2 options: "use" or "not use" in a subset

$$\underset{a_1}{\_}\ \underset{a_2}{\_}\ \underset{a_3}{\_}\ \cdots\ \underset{a_{|A|}}{\_}$$

Example 4.12. (Slides) Jack is so busy that he's always throwing his socks into his top drawer without pairing them. One morning Jack oversleeps. In his haste to get ready for school, (and still a bit sleepy), he reaches into his drawer and pulls out 2 socks. Jack knows that 4 blue socks, 3 green socks, and 2 tan socks are in his drawer.

(a) What are Jack's chances that he pulls out 2 blue socks to match his blue slacks?

$$P(A) = \frac{|A|}{|\Omega|} = \frac{12}{72} = \frac{1}{6}$$

$|A|$: $\underline{4}\times\underline{3}$

$|\Omega|$: $\underline{9}\times\underline{8}$

both are obtained from the multiplication principle.

(b) What are the chances that he pulls out a pair of matching socks?

matching blue socks

matching green socks

matching tan socks

$$\frac{(4\times 3)+(3\times 2)+(2\times 1)}{9\times 8} = \frac{5}{18}$$

Example 4.13. [1, p 29–30] Determine the number of positive integers that are factors of the number

$$3^4 \times 5^2 \times 11^7 \times 13^8.$$

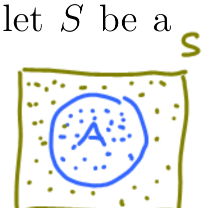
The numbers 3,5,11, and 13 are prime numbers. By the fundamental theorem of arithmetic, each factor is of the form

$$3^i \times 5^j \times 11^k \times 13^\ell,$$

where $0 \le i \le 4$, $0 \le j \le 2$, $0 \le k \le 7$, and $0 \le \ell \le 8$. There are five choices for i, three for j, eight for k, and nine for ℓ. By the multiplication principle, the number of factors is

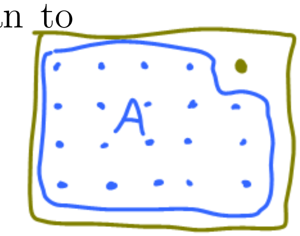
$$5 \times 3 \times 8 \times 9 = 1080.$$

4.14. ***Subtraction Principle***: Let A be a set and let S be a larger set containing A. Then



$$|A| = |S| - |S \setminus A|$$

- When S is the same as Ω, we have $|A| = |\Omega| - |A^c|$
- Using the subtraction principle makes sense only if it is easier to count the number of objects in S and in $S \setminus A$ than to count the number of objects in A.



Example 4.15. *Chevalier de Mere's Scandal of Arithmetic*:

Which is more likely, obtaining at least one six in 4 tosses of a fair dice (event A), or obtaining at least one double six in 24 tosses of a pair of dice (event B)?

$|A| = |\Omega| - |A^c|$

We have

$$P(A) = \frac{6^4 - 5^4}{6^4} = 1 - \left(\frac{5}{6}\right)^4 \approx .518$$

total # of outcomes

and

$$P(B) = \frac{36^{24} - 35^{24}}{36^{24}} = 1 - \left(\frac{35}{36}\right)^{24} \approx .491.$$

Therefore, the first case is more probable.

Remark 1: Probability theory was originally inspired by gambling problems. In 1654, Chevalier de Mere invented a gambling system which bet even money[6] on event B above. However, when he began losing money, he asked his mathematician friend Pascal to analyze his gambling system. Pascal discovered that the Chevalier's system would lose about 51 percent of the time. Pascal became so interested in probability and together with another famous mathematician, Pierre de Fermat, they laid the foundation of probability theory. [U-X-L Encyclopedia of Science]

Remark 2: de Mere originally claimed to have discovered a *contradiction in arithmetic*. De Mere correctly knew that it was advantageous to wager on occurrence of event A, but his experience as gambler taught him that it was not advantageous to wager on occurrence of event B. He calculated $P(A) = 1/6 + 1/6 + 1/6 + 1/6 = 4/6$ and similarly $P(B) = 24 \times 1/36 = 24/36$ which is the same as $P(A)$. He mistakenly claimed that this evidenced a contradiction to the arithmetic law of proportions, which says that $\frac{4}{6}$ should be the same as $\frac{24}{36}$. Of course we know that he could not simply add up the probabilities from each tosses. (By De Meres logic, the probability of at least one head in two tosses of a fair coin would be $2 \times 0.5 = 1$, which we know cannot be true). [21, p 3]

4.16. ***Division Principle (Rule of quotient)***: When a finite set S is partitioned into equal-sized parts of m elements each, there are $\frac{|S|}{m}$ parts.

[6] Even money describes a wagering proposition in which if the bettor loses a bet, he or she stands to lose the same amount of money that the winner of the bet would win.

4.2 Four Kinds of Counting Problems

4.17. Choosing objects from a collection is called **sampling**, and the group/list/sequence of the chosen objects are known as a **sample**. The four kinds of counting problems (and their corresponding formulas) are [9, p 34]:

(a) Ordered sampling of r out of n items with replacement: n^r;

(b) Ordered sampling of $r \leq n$ out of n items without replacement: $(n)_r$;

(c) Unordered sampling of $r \leq n$ out of n items without replacement: $\binom{n}{r}$;

(d) Unordered sampling of r out of n items with replacement: $\binom{n+r-1}{r}$.

- See 4.36 for "bars and stars" argument.

ordered
unordered
sampling
(repetition)
w/ replacement
w/o replacement

Many counting problems can be simplified/solved by realizing that they are equivalent to one of these counting problems.

4.18. Ordered Sampling: Given a set of n distinct items/objects, select a distinct **ordered**[7] sequence (word) of length r drawn from this set.

(a) ***Ordered sampling with replacement***: $\mu_{n,r} = n^r$

- Ordered sampling of r out of n items with replacement.
- The "with replacement" part means "an object can be chosen repeatedly."
- Example: From a deck of n cards, we draw r cards with replacement; i.e., we draw a card, make a note of it, put the card back in the deck and re-shuffle the deck before choosing the next card. How many different sequences of r cards can be drawn in this way? [9, Ex. 1.30] n^r

[7]Different sequences are distinguished by the order in which we choose objects.

n options

$$\underbrace{\_\ \_\ \_\ \_\ \cdots\ \_}_{r \text{ steps}}$$

(b) ***Ordered sampling without replacement***:

$$
\begin{aligned}
(n)_r &= \prod_{i=0}^{r-1}(n-i) = \frac{n!}{(n-r)!} \\
&= \underbrace{n\cdot(n-1)\cdots(n-(r-1))}_{r \text{ terms}}; \quad r \le n
\end{aligned}
$$

- Ordered sampling of $r \le n$ out of n items without replacement.
- The "without replacement" means "once we choose an object, we remove that object from the collection and we cannot choose it again."
- In `Excel`, use `PERMUT(n,r)`.
- Sometimes referred to as "the number of possible **r-permutations** of n distinguishable objects"
- Example: The number of sequences[8] of size r drawn from an alphabet of size n without replacement.

  $(3)_2 = 3 \times 2 = 6$ is the number of sequences of size 2 drawn from an alphabet of size 3 without replacement.

  Suppose the alphabet set is {A, B, C}. We can list all sequences of size 2 drawn from {A, B, C} without replacement:

  A B
  A C
  B A
  B C
  C A
  C B

  same group

  same group

  same group

  For unordered sampling without replacement

  $\# \text{groups} = \frac{(3)_2}{2!} = \frac{6}{2} = 3$

- Example: From a deck of 52 cards, we draw a hand of 5 cards without replacement (drawn cards are not placed back in the deck). How many hands can be drawn in this way?

  _ _ _ _ _

  52 choices, 51 choices, 50, 49, 48

  $52 \times 51 \times 50 \times 49 \times 48 = \frac{52!}{47 \times 46 \times \cdots \times 2 \times 1} = \frac{52!}{47!}$

[8] Elements in a sequence are ordered.

- For integers r, n such that $r > n$, we have $(n)_r = 0$.
- We define $(n)_0 = 1$. (This makes sense because we usually take the empty product to be 1.)
- $(n)_1 = n$
- $(n)_r = (n-(r-1))(n)_{r-1}$. For example, $(7)_5 = (7-4)(7)_4$.
- $(1)_r = \begin{cases} 1, & \text{if r = 1} \\ 0, & \text{if r > 1} \end{cases}$
- Extended definition: The definition in product form

$$(n)_r = \prod_{i=0}^{r-1}(n-i) = \underbrace{n \cdot (n-1) \cdots (n-(r-1))}_{\text{r terms}}$$

can be extended to *any real number* n and a non-negative integer r.

Example 4.19. (Slides) The Seven Card Hustle: Take five red cards and two black cards from a pack. Ask your friend to shuffle them and then, without looking at the faces, lay them out in a row. Bet that them cant turn over three red cards. The probability that they CAN do it is

$$|\Omega| = \underline{7} \times \underline{6} \times \underline{5} = \frac{7!}{4!} = (7)_3 = \frac{7!}{(7-3)!}$$

$$|A| = \underline{5} \times \underline{4} \times \underline{3}$$

$$\frac{5!}{2!} = (5)_3$$

$$P(A) = \frac{5 \times 4 \times 3}{7 \times 6 \times 5} = \frac{2}{7}$$

Definition 4.20. For any integer n greater than 1, the symbol $n!$, pronounced "n ***factorial***," is defined as the product of all positive integers less than or equal to n.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(a) $0! = 1! = 1$

(b) $n! = n(n-1)!$

(c) $n! = \int\limits_0^\infty e^{-t} t^n dt$

(d) Computation:

(i) MATLAB: Use `factorial(n)`. Since double precision numbers only have about 15 digits, the answer is only accurate for $n \leq 21$. For larger n, the answer will have the right magnitude, and is accurate for the first 15 digits.

(ii) Google's web search box built-in calculator: Use `n!`

(e) Approximation: Stirling's Formula [5, p. 52]:

$$n! \approx \sqrt{2\pi n}n^n e^{-n} = \left(\sqrt{2\pi e}\right) e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{2}$$

In some references, the sign $\approx$ is replaced by $\sim$ to emphasize that the ratio of the two sides converges to unity as $n \to \infty$.

4.21. ***Factorial and Permutation***: The number of arrangements (permutations) of $n \geq 0$ distinct items is $(n)_n = n!$.

- Meaning: The number of ways that n distinct objects can be ordered.
  - A special case of ordered sampling without replacement where $r = n$.
- In MATLAB, use `perms(v)`, where v is a row vector of length n, to creates a matrix whose rows consist of all possible permutations of the n elements of v. (So the matrix will contain $n!$ rows and n columns.)

Example 4.22. In MATLAB, `perms([3 4 7])` gives

7 4 3
7 3 4
4 7 3
4 3 7
3 4 7
3 7 4

only two options

_ _ _

3×2×1=6

can use any of the three numbers

only one option

perms([3 3 7])
still lists 6 results

same group

7 3 3
7 3 3
3 7 3
3 3 7
3 3 7
3 7 3

same group

same group

$\frac{6}{2} = 3$ ← # groups

Similarly, `perms('abcd')` gives

$\underline{4}\times\underline{3}\times\underline{2}\times\underline{1} = 4!$
$= 24$

dcba dcab dbca dbac dabc dacb
cdba cdab cbda cbad cabd cadb
bcda bcad bdca bdac badc bacd
acbd acdb abcd abdc adbc adcb

Example 4.23. (Slides) Finger-Smudge on Touch-Screen Devices

Example 4.24. How many people do you need to assemble before the probability is greater than 50% that some two of them have the same birthday (month and day)?
Assumptions:

- Birthdays consist of a month and a day with no year attached.
- Ignore February 29 which only comes in leap years.
- Assume that every day is as likely as any other to be someones birthday.

Probability of coincidence birthday: Probability that there is at least two people who have the same birthdayin a group of r persons:

$$|\Omega| = \underbrace{365 \times 365 \times \cdots \times 365}_{r \text{ times}} = 365^r$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{365 \times 364 \times \cdots \times (365-(r-1))}{(365)^r}$$

$$= \frac{(365)_r}{(365)^r}$$

It is surprising to see, in Figure 2, how quickly the probability approaches 1 as r grows larger.

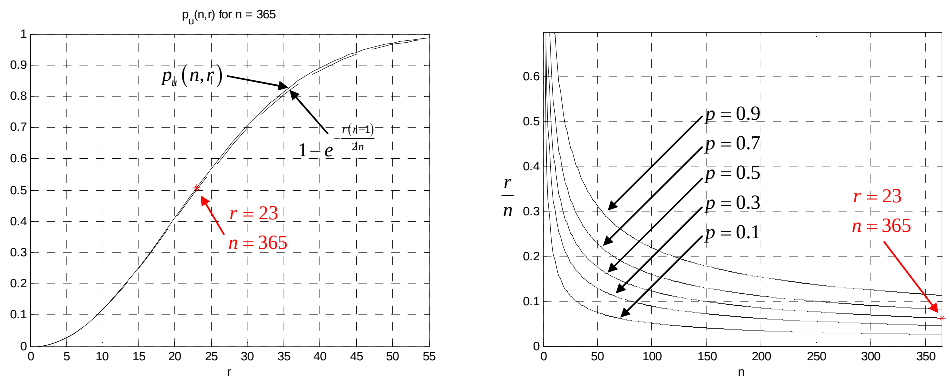


Figure 2: $p_u(n, r)$: The probability of the event that at least one element appears twice in random sample of size r with replacement is taken from a population of n elements.

Birthday Paradox: In a group of 23 randomly selected people, the probability that at least two will share a birthday (assuming birthdays are equally likely to occur on any given day of the year[9]) is about 0.5.

- At first glance it is surprising that the probability of 2 people having the same birthday is so large[10], since there are only 23 people compared with 365 days on the calendar. Some of the surprise disappears if you realize that there are $\binom{23}{2} = 253$ pairs of people who are going to compare their birthdays. [3, p. 9]

Remarks[11]:

- With 88 people, the probability is greater than 1/2 of having three people with the same birthday.

- 187 people gives a probability greater than 1/2 of four people having the same birthday.

[9] In reality, birthdays are not uniformly distributed. In which case, the probability of a match only becomes larger for any deviation from the uniform distribution. This result can be mathematically proved. Intuitively, you might better understand the result by thinking of a group of people coming from a planet on which people are always born on the same day.

[10] In other words, it was surprising that the size needed to have 2 people with the same birthday was so small.

[11] [Rosenhouse, 2009, p 7], [E. H. McKinney, "Generalized Birthday Problem": American Mathematical Monthly, Vol. 73, No.4, 1966, pp. 385-87.]

Example 4.25. Another variant of the birthday coincidence paradox: The group size must be at least 253 people if you want a probability > 0.5 that someone will have the same birthday as you. [3, Ex. 1.13] (The probability is given by $1 - \left(\frac{364}{365}\right)^r$.)

$$P(A) = \frac{|A|}{|\Omega|} = 1 - \frac{|A^c|}{|\Omega|} = 1 - \frac{364 \times 364 \times \cdots \times 364}{(365)^r} = 1 - \frac{(364)^r}{(365)^r}$$

$$= 1 - \left(\frac{364}{365}\right)^r$$

- A naive (but incorrect) guess is that $\lceil 365/2 \rceil = 183$ people will be enough. The "problem" is that many people in the group will have the same birthday, so the number of different birthdays is smaller than the size of the group.

- On late-night television's The Tonight Show with Johnny Carson, Carson was discussing the birthday problem in one of his famous monologues. At a certain point, he remarked to his audience of approximately 100 people: "Great! There must be someone here who was born on my birthday!" He was off by a long shot. Carson had confused two distinctly different probability problems: (1) the probability of one person out of a group of 100 people having the same birth date as Carson himself, and (2) the probability of any two or more people out of a group of 101 people having birthdays on the same day. [21, p 76]

4.26. Now, let's revisit ordered sampling of r out of n different items without replacement. One way to look at the sampling is to first consider the $n!$ permutations of the n items. Now, use only the first r positions. Because we do not care about the last $n - r$ positions, we will group the permutations by the first r positions. The size of each group will be the number of possible permutations of the $n - r$ items that has not already been used in the first r

positions. So, each group will contain $(n-r)!$ members. By the division principle, the number of groups is $n!/(n-r)!$.

4.27. The number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of the second type, n_3 are of the third type, $\ldots$, and n_r are of the rth type is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

Example 4.28. The number of permutations of AABC

$$\frac{4!}{2} = 4 \times 3 = 12$$

Example 4.29. The number of permutations of $A_1A_2A_3BC$

$A_1BCA_2A_3$
$A_2BCA_1A_3$
$A_3BCA_1A_2$
$\vdots$

$$\frac{5!}{3!} = 5 \times 4 = 20$$

Example 4.30. The number of permutations of $A_1A_2B_1B_2C_1C_2C_3$

$$\left(\frac{7!}{2!}\right) \times \frac{1}{2!} \times \frac{1}{3!} = \frac{7!}{2!2!3!} = \binom{7}{2,2,3}$$

Example 4.31. Bar Codes: A part is labeled by printing with four thick lines, three medium lines, and two thin lines. If each ordering of the nine lines represents a different label, how many different labels can be generated by using this scheme?

$$\frac{9!}{4!\,3!\,2!} = 1,260$$

4.32. *Binomial coefficient*:

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}$$

(a) Read "n choose r".

(b) Meaning:

(i) **Unordered sampling** of $r \le n$ out of n distinct items **without replacement**

$\frac{n!}{r!(n-r)!}$

Ex. $n = 5$
choose 3 letters
from 5 letters
A B C D E

$$\frac{\underline{5} \times \underline{4} \times \underline{3}}{3!} = \frac{(5)_3}{3!}$$

ordered sampling
without replacement
↓
unordered

$\frac{(n)_r}{r!}$

(ii) The number of subsets of size r that can be formed from a set of n elements (without regard to the order of selection).

(iii) The number of combinations of n objects selected r at a time.

(iv) the number of r-**combinations** of n objects.

(v) The number of (unordered) sets of size r drawn from an alphabet of size n without replacement.

(c) Computation:

(i) `MATLAB`:

- `nchoosek(n,r)`, where n and r are nonnegative integers, returns $\binom{n}{r}$.
- `nchoosek(v,r)`, where v is a row vector of length n, creates a matrix whose rows consist of all possible combinations of the n elements of v taken r at a time. The matrix will contains $\binom{n}{r}$ rows and r columns.

- Example: `nchoosek('abcd',2)` gives

  ab
  ac
  ad
  bc
  bd
  cd

(ii) `Excel`: `combin(n,r)`

(iii) `Mathcad`: `combin(n,r)`

(iv) `Maple`: $\binom{n}{r}$

(v) Google's web search box built-in calculator: `n choose r`

(d) Reflection property: $\binom{n}{r} = \binom{n}{n-r}$.

(e) $\binom{n}{n} = \binom{n}{0} = 1$.

(f) $\binom{n}{1} = \binom{n}{n-1} = n$.

(g) $\binom{n}{r} = 0$ if $n < r$ or r is a negative integer.

(h) $\max_r \binom{n}{r} = \binom{n}{\lfloor \frac{n+1}{2} \rfloor}$.

Example 4.33. In bridge, 52 cards are dealt to four players; hence, each player has 13 cards. The order in which the cards are dealt is not important, just the final 13 cards each player ends up with. How many different bridge games can be dealt? (Answer: 53,644,737,765,488,792,839,237,440,000)

Technique # 1

$$\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13}$$

$$= \frac{52!}{\cancel{39!}\,13!} \times \frac{\cancel{39!}}{\cancel{26!}\,13!} \times \frac{\cancel{26!}}{13!\,13!} \times \frac{\cancel{13!}}{\cancel{13!}\,0!} = \frac{52!}{(13!)^4} \approx 5 \times 10^{28}$$

Technique # 2

$$\underbrace{\_\ \_\ \_\ \_}_{\text{13 positions for player 1}}\Big[\underbrace{\_\ \_}_{\text{13 positions for player 2}}\Big]\cdots\Big[\underbrace{\_}_{\text{for player 4}}\Big] \qquad \frac{52!}{13!\,13!\,13!\,13!}$$

4.34. ***Unordered sampling with replacement***: There are n items. We sample r out of these n items with replacement. Because the order in the sequences is not important in this kind of sampling, two samples are distinguished by the number of each item in the sequence. In particular, suppose r letters are drawn with replacement from a set $\{a_1, a_2, \ldots, a_n\}$. Let x_i be the number of a_i in the drawn sequence. Because we sample r times, we know that, for every sample, $x_1 + x_2 + \cdots x_n = r$ where the x_i are non-negative integers. By the bars-and-stars argument below, there are $\binom{n+r-1}{r}$ possible unordered samples with replacement.

Example 4.35. Suppose the items are four different letters A,B,C,D ($n = 4$). We sample $r = 8$ out of these n items with replacement.

Assume unordered sampling

A sample would be BBCCABCA

This is the same as AABBBCCC.

The only important information here (unordered sampling) is the # of each type.

Let

x_A = # A in the sample

x_B = # B " "

x_C = # C " "

x_D = # D " "

We know that

$x_A + x_B + x_C + x_D = 8$ ← r

Example 4.36. The **bars-and-stars argument**: Find all non-negative integers x_1, x_2, x_3 such that

$$x_1 + x_2 + x_3 = 3.$$

$\frac{11!}{8!\,3!} = 165$

* * *
1 1 1

| * | * *
0 1 2

$0+0+3$	||111
$0+1+2$	|1|11
$0+2+1$	|11|1
$0+3+0$	|111|
$1+0+2$	1||11
$1+1+1$	_____
$1+2+0$	_____
$2+0+1$	_____
$2+1+0$	_____
$3+0+0$	_____

$\binom{5}{2} = \binom{5}{3} = \frac{5!}{2!\,3!} = 10$

We see that any such configuration stands for a solution to the equation, and any solution to the equation can be converted to such a walls-ones series. So we've established a bijection between the solutions to our equation and the configurations of two walls and three ones. So our problem reduces to "in how many ways can we place two walls and three ones in five places?" We can do this in $\binom{5}{2}$ ways. So the number of solutions to our equation is $\binom{5}{2} = 10$.

Example 4.37. Consider the equation

$$x_1 + x_2 + x_3 + \cdots + x_{10} = 15$$

where $x_1, x_2, x_3, \ldots, x_{10}$ are nonnegative integers. How many solutions does this equation have?

9 walls

$$\overbrace{|\,|\,\underbrace{1\;1\,|\,1\;1\,|\,1\,|\,1\;1}_{15 \text{ 1's}}}$$

$$\binom{24}{15} = \binom{24}{9} = \frac{24!}{9!\,15!} = 1,307,000$$

4.38. Summary and Extension: There are $\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$ distinct n-tuples $(x_1, x_2, \ldots, x_n)$ of nonnegative integers such that $x_1 + x_2 + \cdots + x_n = r$.

- We use $n - 1$ walls to separate r 1's.
- This is the same as the number of ways to place r indistinguishable balls into n labeled urns.

(a) Suppose we further require that the x_i are strictly positive ($x_i \geq 1$), then there are $\binom{r-1}{n-1}$ solutions.

(b) ***Extra Lower-bound Requirement***: Suppose we further require that $x_i \geq a_i$ where the a_i are some given nonnegative integers, then the number of solution is

$$\binom{r - (a_1 + a_2 + \cdots + a_n) + n - 1}{n - 1}.$$

Note that here we work with equivalent problem: $y_1 + y_2 + \cdots + y_n = r - \sum_{i=1}^{n} a_i$ where $y_i \geq 0$.

Example 4.39. Suppose words that are anagrams are considered the same. How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement?

Observe that since anagrams are considered the same, the feature of interest is how many times each letter appears in the word (ignoring the order in which the letters appear). To translate this into a stars-and-bars problem, we consider writing "5" as a sum of 26 integers $n_A, n_B, \ldots, n_Z$ where n_A is the number of times letter A is chosen, n_B is the number of times letter B is chosen, etc.

Then by (4.38), the number of 5-letter words is

$$\binom{5+26-1}{5} = \binom{30}{5} = 142,506.$$

4.40. For the "unordered sampling with replacement" calculation, it is tempting to start with the formula n^r for the "ordered sampling with replacement" case and then change to the "unordered sampling" case by $\times\frac{1}{r!}$ via the division principle. (This was, after all, the technique that we used back when we considered "sampling without replacement" in 4.32.

However, turn out that the same technique can't be applied here. This is because one key requirement for applying the division principle is that each group should contain the same number of member. When we did the "sampling without replacement", we are guaranteed to have r distinct objects. However, when the sampling is with replacement, some objects may be chosen more than once. We have already seen, in 4.27, that the number of possibilities when permuting r objects that are not all distinct is not $r!$. More importantly, the numbers of possibilities are different depending on how many repeated objects in each type. So, there are various group sizes invalidating the application of division principle.

For example, suppose we have two object types: A and B. Let's select two objects using "unordered sampling with replacement". There are three possibilities: AA, AB, and BB. (Note that BA is the same as AB because the sampling is unordered.) If we start with "ordered sampling with replacement", we have four possibilities: AA, AB, BA, and BB. Grouping these possibilities using permutation, we have three groups: {AA}, {AB,BA}, {BB}. As mentioned earlier, the group sizes are not the same and therefore we can't directly apply the division principle.

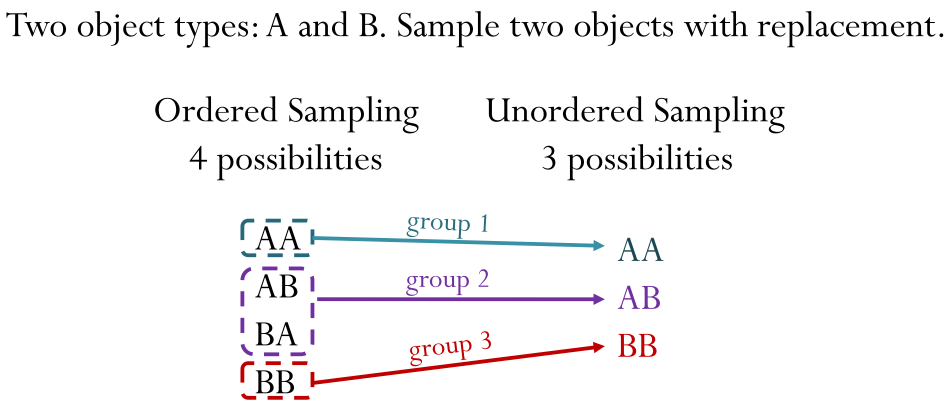


Figure 3: Division principle can't be applied easily to convert the formula for "ordered sampling with replacement" to the formula for "unordered sampling with replacement."

4.3 Binomial Theorem and Multinomial Theorem

4.41. ***Binomial theorem***: Sometimes, the number $\binom{n}{r}$ is called a ***binomial coefficient*** because it appears as the coefficient of $x^r y^{n-r}$ in the expansion of the binomial $(x+y)^n$. More specifically, for any positive integer n, we have,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \tag{3}$$

For example,

$$\begin{aligned}(x+y)^3 &= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3 \\ &= x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

To see why this is true, we will first try to directly multiply the sums. However, to keep track of the variables, let's first treat them as distinct as shown in Figure 4. Under such consideration, observe that expansion converts a product of sums into a sum of products. Each resulting product contains a term in the first sum, a term in the second sum, and a term in the third sum. All the products have unit coefficient. Product terms of the form x^3, x^2y, xy^2, and y^3 arise after we try to convert x_1, x_2, x_3 back to x and y_1, y_2, y_3 back to y. Some product terms are the same and hence can be combined resulting in the non-unity coefficients.

$$(x_1+y_1)\times(x_2+y_2)$$
$$=x_1x_2+x_1y_2+y_1x_2+y_1y_2$$
$$(x_1+y_1)\times(x_2+y_2)\times(x_3+y_3)$$
$$=x_1x_2x_3+x_1x_2y_3+x_1y_2x_3+x_1y_2y_3+y_1x_2x_3+y_1x_2y_3+y_1y_2x_3+y_1y_2y_3$$
$$\Big\Downarrow \quad \begin{aligned} x_1 &= x_2 = x_3 = x \\ y_1 &= y_2 = y_3 = y \end{aligned}$$
$$(x+y)\times(x+y)$$
$$=xx+xy+yx+yy=x^2+2xy+y^2$$
$$(x+y)\times(x+y)\times(x+y)$$
$$=xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$$
$$=x^3+3x^2y+3xy^2+y^3$$

Figure 4: Binomial expansion: when treating all variables as distinct, in the sum of products, we have a term from each sum that are multiplied in the original expression.

The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added.

To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two terms). This can be done in $\binom{3}{1}$ ways. Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums in the product, and this can be done in exactly one way. Consequently. it follows that

$$(x+y)^3 = x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + y^3.$$

Now, let's state a combinatorial proof of the binomial theorem (3). The terms in the product when it is expanded are of the form $x^r y^{n-r}$ for $r = 0, 1, 2, \ldots, n$. To count the number of terms of the form $x^r y^{n-r}$, note that to obtain such a term it is necessary to choose r xs from the n sums (so that the other $n-r$ terms in the product are ys). Therefore. the coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$.

4.42. From (3), if we let $x = y = 1$, then we get another important identity:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n. \tag{4}$$

One interpretation of (4) is to think about the size of a power set. Consider a set A with n (distinct) elements. We have seen in 4.32 that A has $\binom{n}{r}$ subsets of size r. Therefore, the sum on the left in (4) is trying to count the number of all possible subsets of A. In other words, the sum gives the size of the power set of A. In Example 4.11, we have already shown that this number is $2^{|A|} = 2^n$. This reasoning gives (4) without knowing the binomial theorem.

Definition 4.43. ***Multinomial Counting***: The ***multinomial coefficient***

$$\binom{n}{n_1,\ n_2,\ \ldots,\ n_r}$$

is defined as

$$\prod_{i=1}^{r}\binom{n-\sum\limits_{k=0}^{i-1}n_k}{n_i} = \binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdots\binom{n_r}{n_r}$$
$$= \frac{n!}{\prod\limits_{i=1}^{r} n_i!}.$$

We have seen this before in (4.27). It is the number of ways that we can arrange $n = \sum\limits_{i=1}^{r} n_i$ tokens when having r types of symbols and n_i indistinguishable copies/tokens of a type i symbol.

4.44. ***Multinomial Theorem***:

$$(x_1+\ldots+x_r)^n = \sum \frac{n!}{i_1!i_2!\cdots i_r!}x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r},$$

where the sum ranges over all ordered r-tuples of integers $i_1,\ldots,i_r$ satisfying the following conditions:

$$i_1\geq 0,\ldots,i_r\geq 0,\quad i_1+i_2+\cdots+i_r=n.$$

When $r=2$ this reduces to the binomial theorem.

Example 4.45. Find the coefficient of x^3yz in the expansion of $(x+y+z)^5$

$$\binom{5}{3}\binom{2}{1}\binom{1}{1} = \frac{5!}{3!\,\cancel{2!}}\ \frac{\cancel{2!}}{1!\,1!}\ \frac{\cancel{1!}}{\cancel{1!}\,0!} = \frac{5!}{3!\,1!\,1!}$$

$$\underbrace{\_\ \_\ \_}_{x}\ \underset{y}{\_}\ \underset{z}{\_} \qquad \frac{5!}{3!\,1!\,1!}$$

ECS315 2018/1 Part I.3 Dr.Prapun

4.46. Further reading on combinatorial ideas: the pigeon-hole principle, inclusion-exclusion principle, generating functions and recurrence relations, and flows in networks.

4.4 Famous Example: Galileo and the Duke of Tuscany

Example 4.47. When you toss three dice, the chance of the sum being 10 is greater than the chance of the sum being 9.

- The Grand Duke of Tuscany "ordered" Galileo to explain a paradox arising in the experiment of tossing three dice [2]:

  > "Why, although there were an equal number of 6 partitions of the numbers 9 and 10, did experience state that the chance of throwing a total 9 with three fair dice was less than that of throwing a total of 10?"

- Partitions of sums 11, 12, 9 and 10 of the game of three fair dice:

1+4+6=11	1+5+6=12	3+3+3=9	1+3+6=10
2+3+6=11	2+4+6=12	1+2+6=9	1+4+5=10
2+4+5=11	3+4+5=12	1+3+5=9	2+2+6=10
1+5+5=11	2+5+5=12	1+4+4=9	2+3+5=10
3+3+5=11	3+3+6=12	2+2+5=9	2+4+4=10
3+4+4=11	4+4+4=12	2+3+4=9	3+3+3=10

The partitions above are not equivalent. For example, from the addenda 1, 2, 6, the sum 9 can come up in $3! = 6$ different ways; from the addenda 2, 2, 5, the sum 9 can come up in $\frac{3!}{2!1!} = 3$ different ways; the sum 9 can come up in only one way from 3, 3, 3.

- ***Remarks***: Let X_i be the outcome of the ith dice and S_n be the sum $X_1 + X_2 + \cdots + X_n$.

  (a) $P[S_3 = 9] = P[S_3 = 12] = \frac{25}{6^3} < \frac{27}{6^3} = P[S_3 = 10] = P[S_3 = 11]$. Note that the difference between the two probabilities is only $\frac{1}{108}$.

  (b) The range of S_n is from n to $6n$. So, there are $6n-n+1 = 5n+1$ possible values.

  (c) The pmf of S_n is symmetric around its expected value at $\frac{n+6n}{2} = \frac{7n}{2}$.

    ◦ $P[S_n = m] = P[S_n = 7n - m]$.

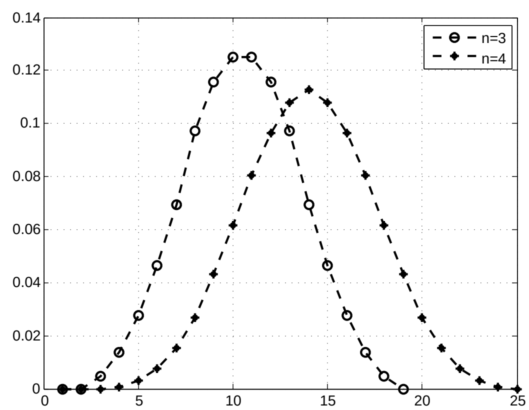


Figure 5: pmf of S_n for $n = 3$ and $n = 4$.

4.5 Application: Success Runs

Example 4.48. We are all familiar with "success runs" in many different contexts. For example, we may be or follow a tennis player and count the number of consecutive times the player's first serve is good. Or we may consider a run of forehand winners. A basketball player may be on a "**hot streak**" and hit his or her shots perfectly for a number of plays in row.

In all the examples, whether you should or should not be amazed by the observation depends on a lot of other information. There may be perfectly reasonable explanations for any particular success run. But we should be curious as to whether randomness could also be a perfectly reasonable explanation. Could the hot streak of a player simply be a snapshot of a random process, one that we particularly like and therefore pay attention to?

In 1985, cognitive psychologists Amos Taversky and Thomas Gilovich examined[12] the shooting performance of the Philadelphia 76ers, Boston Celtics and Cornell University's men's basketball team. They sought to discover whether a player's previous shot had any predictive effect on his or her next shot. Despite basketball fans' and players' widespread belief in hot streaks, the researchers found no support for the concept. (No evidence of nonrandom behavior.) [14, p 178]

4.49. Academics call the mistaken impression that a random streak is due to extraordinary performance the **hot-hand fallacy**. Much of the work on the hot-hand fallacy has been done in the context of sports because in sports, performance is easy to define and measure. Also, the rules of the game are clear and definite, data are plentiful and public, and situations of interest are replicated repeatedly. Not to mention that the subject gives academics a way to attend games and pretend they are working. [14, p 178]

Example 4.50. Suppose that two people are separately asked to toss a fair coin 120 times and take note of the results. Heads is noted as a "one" and tails as a "zero". The following two lists of compiled zeros and ones result

1 1 0 0 1 0 0 1 0 1 1 0 0 1 0 0 0 1 1 0 1 0 1 0 0 1 1 0 1 0
0 1 0 1 0 1 1 0 1 1 0 0 1 1 0 1 1 1 0 1 0 0 1 0 0 1 1 0 1 0
0 1 1 0 1 0 0 1 1 0 1 0 1 1 0 0 1 1 1 0 0 1 0 1 0 1 0 0 0 1
0 1 0 1 0 1 0 1 0 1 1 0 0 1 0 0 1 0 1 1 0 0 1 0 0 1 1 0 1 1

and

[12] "The Hot Hand in Basketball: On the Misperception of Random Sequences"

1 1 1 0 0 0 1 1 1 0 1 0 1 1 1 1 1 1 0 1 0 0 0 1 1 0 0 1 1 0
1 0 1 0 0 0 1 1 0 1 0 0 1 1 1 0 1 0 0 0 0 1 0 1 1 1 0 1 1 0
0 1 1 1 0 1 1 0 0 1 1 1 1 1 1 0 1 1 0 1 0 1 1 1 0 0 0 0 0 0
0 0 1 1 0 1 1 1 0 1 1 1 1 0 1 1 1 1 0 1 0 1 1 0 1 1 0 1 0 1

One of the two individuals has cheated and has fabricated a list of numbers without having tossed the coin. Which list is more likely be the fabricated list? [21, Ex. 7.1 p 42–43]

The answer is later provided in Example 4.56.

Definition 4.51. A **run** is a sequence of more than one consecutive identical outcomes, also known as a **clump**.

Definition 4.52. Let R_n represent the length of the longest run of heads in n independent tosses of a fair coin. Let $\mathcal{A}_n(x)$ be the set of (head/tail) sequences of length n in which the longest run of heads does not exceed x. Let $a_n(x) = \|\mathcal{A}_n(x)\|$.

Example 4.53. If a fair coin is flipped, say, three times, we can easily list all possible sequences:

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

and accordingly derive:

x	$P[R_3 = x]$	$a_3(x)$
0	1/8	1
1	4/8	4
2	2/8	7
3	1/8	8

4.54. Consider $a_n(x)$. Note that if $n \le x$, then $a_n(x) = 2^n$ because any outcome is a favorable one. (It is impossible to get more than three heads in three coin tosses). For $n > x$, we can partition $\mathcal{A}_n(x)$ by the position k of the first tail. Observe that k must be $\le x + 1$ otherwise we will have more than x consecutive heads in the sequence which contradicts the definition of $\mathcal{A}_n(x)$. For each $k \in \{1, 2, \dots, x + 1\}$, the favorable sequences are in the form

$$\underbrace{\text{HH}\dots\text{H}}_{k-1 \text{ heads}}\text{T}\ \underbrace{\text{XX}\dots\text{X}}_{n-k \text{ positions}}$$

where, to keep the sequences in $\mathcal{A}_n(x)$, the last $n-k$ positions[13] must be in $\mathcal{A}_{n-k}(x)$. Thus,

$$a_n(x) = \sum_{k=1}^{x+1} a_{n-k}(x) \text{ for } n > x.$$

In conclusion, we have

$$a_n(x) = \begin{cases} \sum_{j=0}^{x} a_{n-j-1}(x), & n > x, \\ 2^n & n \le x \end{cases}$$

[20]. The following MATLAB function calculates $a_n(x)$

```
function a = a_nx(n,x)
a = [2.^(1:x) zeros(1,n-x)];
a(x+1) = 1+sum(a(1:x));
for k = (x+2):n
    a(k) = sum(a((k-1-x):(k-1)));
end
a = a(n);
```

4.55. Similar technique can be used to construct $\mathcal{B}_n(x)$ defined as the set of sequences of length n in which the longest run of heads and the longest run of tails do not exceed x. To check whether a sequence is in $\mathcal{B}_n(x)$, first we convert it into sequence of S and D by checking each adjacent pair of coin tosses in the original sequence. S means the pair have same outcome and D means they are different. This process gives a sequence of length $n-1$. Observe that a string of $x-1$ consecutive S's is equivalent to a run of length x. This put us back to the earlier problem of finding $a_n(x)$ where the roles of H and T are now played by S and D, respectively. (The length of the sequence changes from n to $n-1$ and the max run length is $x-1$ for S instead of x for H.) Hence, $b_n(x) = \|\mathcal{B}_n(x)\|$ can be found by

$$b_n(x) = 2a_{n-1}(x-1)$$

[20].

[13] Strictly speaking, we need to consider the case when $n = x+1$ separately. In such case, when $k = x+1$, we have $\mathcal{A}_0(x)$. This is because the sequence starts with x heads, then a tail, and no more space left. In which case, this part of the partition has only one element; so we should define $a_0(x) = 1$. Fortunately, for $x \ge 1$, this is automatically satisfied in $a_n(x) = 2^n$.

Example 4.56. Continue from Example 4.50. We can check that in 120 tosses of a fair coin, there is a very large probability that at some point during the tossing process, a sequence of five or more heads or five or more tails will naturally occur. The probability of this is

$$\frac{2^{120} - b_{120}(4)}{2^{120}} \approx 0.9865.$$

0.9865. In contrast to the second list, the first list shows no such sequence of five heads in a row or five tails in a row. In the first list, the longest sequence of either heads or tails consists of three in a row. In 120 tosses of a fair coin, the probability of the longest sequence consisting of three or less in a row is equal to

$$\frac{b_{120}(3)}{2^{120}} \approx 0.000053,$$

which is extremely small indeed. Thus, the first list is almost certainly a fake. Most people tend to avoid noting long sequences of consecutive heads or tails. Truly random sequences do not share this human tendency! [21, Ex. 7.1 p 42–43]